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SELF-SIMILAR MOTION OF FLUID UNDER THE ACTION OF SURFACE TENSION

F. L. Chernous'ko

ABSTRACT

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Investigation of the behavior of a weightless fluid with a free surface set in motion from a state of rest by the force of surface tension. A particular case of the problem of self-similarity is formulated and solved.

Author

1. Let us examine the plane flow of an ideal, incompressible, weightless liquid having the density ρ. Let x,y be the rectangular Cartesian coordinates in the flow plane. At the moment of time $t = 0$, the liquid is at rest and occupies the wedge (Figure 1) with the angle of opening α, which is bounded by the free surface $y = 0$ and the solid wall $y = -x\,tg\alpha$. The coefficient of surface tension σ at the free boundary and the contact angle γ at the boundary between the liquid and the wall (Figure 1) are assumed to be constant. If $\gamma \neq \alpha$, then for $t > 0$ the liquid is set in motion, and the flow will have a potential. This type of flow can develop when the surface tension is suddenly "switched on", and also in the following case, for example. For $t < 0$, let the liquid /54*

Note: Numbers in the margin indicate pagination in the original foreign text.

be at rest in the field of gravity. The free surface thus differs significantly from the plane y = 0 only in the region close to the wall, where a meniscus is formed (Ref. 1). The greater the ratio between the force of gravity and the force of surface tension, the smaller will be the meniscus. At the moment t = 0, let the force of gravity instantaneously become zero. Then a flow develops for t > 0, which will be close to the self-similar flow being examined here, if the dimensions of the initial meniscus are small as compared with the scale of the process in which we are interested (i.e., if the force of gravity is sufficiently large for t < 0).

The velocity potential $\varphi^0(x,y,t)$ satisfies the Laplace equation in the region of flow and the condition of rest at the wall (the indices below designate the partial derivatives)

$$\varphi_{xx}^\circ + \varphi_{yy}^\circ = 0, \qquad \varphi_y^\circ + \varphi_x^\circ \operatorname{tg}\alpha = 0 \text{ for } y = -x \operatorname{tg}\alpha \tag{1.1}$$

The pressure p in the liquid having the free surface $y = f^0(x,t)$ is connected by the following relationship with the constant pressure p_0 outside of the liquid (Ref. 1):

$$p = p_0 - \sigma K, \qquad K = \pm f_{xx}^\circ (1 + f_x^{\circ 2})^{-3/2} \tag{1.2}$$

K represents the curvature of the free surface. The upper sign in the formula (1.2), and also in (1.7), must be taken when the liquid lies below the free surface (just as in Figure 1), and the lower sign - in the opposite case.

Taking into account the formula for p, we can write the Cauchy-Lagrange /55
integral for points on the free surface, as well as the kinematic condition

$$\varphi_t^\circ + {}^1/_2\,(\nabla\varphi^\circ)^2 - \sigma K/\rho = 0, \quad f_t^\circ + f_x^\circ\varphi_x^\circ - \varphi_y^\circ = 0 \text{ for } y = f^\circ(x, t) \tag{1.3}$$

At the point at which the free boundary contacts the wall, we have

$$f_x^\circ(x, t) = \operatorname{tg}(\gamma - \alpha) \text{ for } y = f^\circ(x, t) = -x \operatorname{tg}\alpha \tag{1.4}$$

The initial conditions and conditions at infinity have the following form

$$\varphi^\circ(x, y, 0) = f^\circ(x, 0) \equiv 0, \quad \varphi^\circ, f^\circ \to 0 \text{ for } x, y \to \infty \tag{1.5}$$

The problem (1.1) - (1.5) of determining the function φ^0, f^0 will be self-similar: it contains two dimensional parameters σ, ρ having the dimensions $[\sigma] = MT^{-2}$, $[\rho] = ML^{-3}$. Let us introduce the dimensionless, independent variables ξ, η and the dimensionless desired functions ϕ, f

$$x = \left(\frac{\sigma t^2}{\rho}\right)^{1/3}\xi, \quad y = \left(\frac{\sigma t^2}{\rho}\right)^{1/3}\eta, \quad \varphi^\circ = \left(\frac{\sigma^2 t}{\rho^2}\right)^{1/3}\varphi(\xi, \eta), \quad f^\circ = \left(\frac{\sigma t^2}{\rho}\right)^{1/3} f(\xi) \tag{1.6}$$

Changing to new variables in equations (1.1) - (1.5), according to (1.6), we obtain the boundary value problem for the function φ, f

$$\begin{gathered}
\varphi_{\xi\xi} + \varphi_{\eta\eta} = 0, \quad \varphi_\eta + \varphi_\xi \operatorname{tg}\alpha = 0 \text{ for } \eta = -\xi \operatorname{tg}\alpha \\
{}^1/_3\,\varphi - {}^2/_3\,(\xi\varphi_\xi + \eta\varphi_\eta) + {}^1/_2\,(\nabla\varphi)^2 \mp f''\,(1 + f'^2)^{-3/2} = 0 \\
{}^2/_3\,(f - \xi f') + f'\varphi_\xi - \varphi_\eta = 0 \text{ for } \eta = f(\xi) \\
f' = \operatorname{tg}(\gamma - \alpha) \text{ for } f(\xi) = -\xi \operatorname{tg}\alpha \\
\varphi(\xi, \eta) \to 0, \quad f(\xi) \to 0 \text{ for } \xi, \eta \to \infty
\end{gathered} \tag{1.7}$$

The dashed line designates the derivative with respect to ξ. A non-linear boundary value problem (1.7) is formulated for the region (Figure 1) bounded by the line $\eta = -\xi \operatorname{tg}\alpha$ and the unknown curve $\eta = f(\xi)$. Self-similar, axi-symmetric flow can be examined in a similar manner.

2. Problem (1.7) can be linearized, if the angles γ and α are similar to each other, i.e., $\gamma - \alpha = \varepsilon, |\varepsilon| \ll 1$. For purposes of linearization, we assume that the functions φ, f and their derivatives are of a small order ε, and the conditions on the line $\eta = 0$ are removed from the unknown boundary $\eta = f(\xi)$.

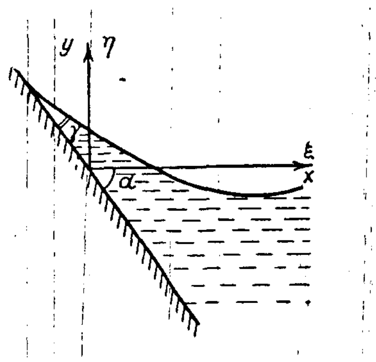

Figure 1

We must select the upper of the two signs in (1.7), since for $|\varepsilon| \ll 1$ the disturbances of the free boundary are small, and the liquid lies below the free surface.

For purposes of definition, let us set $\alpha = {}^1/_2\pi$. Then the linearized boundary value problem can be reduced to determining the function $\varphi(\xi,\eta)$, which is harmonic in the quadrant $\xi>0, \eta<0$, as well as the function $f(\xi)$ according to the condition

$$\begin{gathered} {}^1/_3\varphi - {}^2/_3\xi\varphi_\xi - f'' = 0, \quad {}^2/_3(f - \xi f') - \varphi_\eta = 0 \quad \text{for } \eta = 0 \\ \varphi_\xi = 0 \quad \text{for } \xi = 0, \qquad f'(0) = \varepsilon \\ \varphi(\xi,\eta) \to 0, \quad f(\xi) \to 0 \quad \text{for } \xi, \eta \to \infty \end{gathered} \tag{2.1}$$

Differentiating the second condition (2.1) with respect to ξ, we /56 exclude f'' from the condition for $\eta = 0$, and we obtain the boundary condition for φ in the form

$$\begin{gathered}\varphi_{\xi\eta} - {}^4/_9\,\xi^2\varphi_\xi + {}^2/_9\,\xi\varphi = 0 \text{ for } \eta = 0\\ \varphi_\xi = 0 \text{ for } \xi = 0, \qquad \varphi \to 0 \text{ for } \xi, \eta \to \infty\end{gathered} \tag{2.2}$$

The uniform linear boundary value problem (2.2) contains the second derivative $\varphi_{\xi\eta}$ in the boundary condition, and does not pertain to a number of boundary value problem types which have been studied. As will be shown, this problem has a one parametric set of solutions, and a single solution is obtained by the condition $f'(0) = \varepsilon$.

Let us introduce the complex variable $z = \xi+i\eta$ and the complex potential $w = \varphi+i\psi$, where ψ is the harmonic function conjugate to φ. Without loss of generality, we can assume that $\psi = 0$ on the basis of (2.2) for $\xi = 0$. Therefore, the analytical function w(z) can be continued by symmetry in the entire lower half-plane $\eta<0$. As follows from (2.2), on the real axis we obtain (dashed line designates derivative with respect to z)

$$\operatorname{Re}\,(iw'' - {}^4/_9\,z^2w' + {}^2/_9\,zw) = 0 \quad \text{for } \eta = 0, \tag{2.3}$$

Let us make the single assumption that w strives to 0 as the dipole potential for $z \to \infty$, i.e., $w = 0(z^{-1})$, $w' = 0(z^{-2})$, $w'' = 0(z^{-3})$ for $z \to \infty$. This assumption is substantiated by the fact that a single solution which has such an asymptotic behavior for $z \to \infty$ will be formulated below.

The function under the sign Re in (2.3) is analytic for $\eta < 0$, and - in view of the given assumption - is bounded at infinity. From (2.3) we then have

$$iw'' - {}^{4}/_{9}z^{2}w' + {}^{2}/_{9}zw = iC \tag{2.4}$$

Here C is still an arbitrary real constant.

Thus, the boundary value problem (2.1) can be reduced to determining a solution of the ordinary linear differential equation (2.4) for the given asymptotic behavior $w = O(z^{-1})$ at infinity.

When the function $w(z) = \varphi + i\psi$ is found, the form of the free surface $f(\xi)$ can be determined from the second condition (2.1), which represents a linear equation of the first order for f. Taking $\varphi_\eta = -\psi_\xi$ into account, we can write a solution of this equation which satisfies the condition $f(\infty) = 0$:

$$f(\xi) = -\frac{3\xi}{2}\int_{\xi}^{\infty}\frac{\psi_\xi(x,0)}{x^2}dx \tag{2.5}$$

Let us differentiate (2.5) and equate $f'(0) = \varepsilon$

$$f'(\xi) = -\frac{3}{2}\int_{\xi}^{\infty}\frac{\psi_\xi(x,0)}{x^2}dx + \frac{3\psi_\xi(\xi,0)}{2\xi} = \frac{3}{2}\int_{\xi}^{\infty}\frac{\psi_\xi(\xi,0)-\psi_\xi(x,0)}{x^2}dx$$
$$\frac{3}{2}\int^{\infty}\frac{\psi_\xi(0,0)-\psi_\xi(x,0)}{x^2}dx = \varepsilon \tag{2.6}$$

The integral (2.6) converges for $x = \infty$ in view of the asymptotic /57 behavior $w = O(z^{-1})$ and $\psi_\xi = O(\xi^{-2})$. From the condition of symmetry $\psi(0,\eta) = 0$, it follows that $\psi(\xi,0)$ is odd, and $\psi_\xi(\xi,0)$ is an even function of ξ. This means that $\psi_\xi(0,0)-\psi_\xi(x,0) = O(x^2)$ for $x \to 0$, and the integral (2.6) converges for $x = 0$. The condition (2.6) can be used to determine the constant C from (2.4), which enters into w with a multiplier. It can be readily shown that for $f(\xi)$ the condition expressing conservation of the liquid mass can be fulfilled from (2.5):

$$\int_0^\infty f(\xi)\, d\xi = 0$$

3. The particular solution w^0 of the non-homogeneous equation (2.4) and the linearly-independent, special solutions w^1, w^2 of the corresponding (2.4) homogeneous equation can be written in the following form:

$$w_0 = C \sum_{k=0}^{\infty} a_k z^{3k+2}, \qquad w_1 = \sum_{k=0}^{\infty} a_k' z^{3k}, \qquad w_2 = \sum_{k=0}^{\infty} a_k'' z^{3k+1}$$

Substituting each of these series in the equation and setting the coefficients equal for powers of z, we obtain the recurrent relationships (the coefficients a_0', a_0'' are arbitrary):

$$a_0 = \frac{1}{2}, \qquad \frac{a_k}{a_{k-1}} = \frac{2(-i)(2k-1)}{3(3k+1)(3k+2)}, \qquad \frac{a_k'}{a_{k-1}'} = \frac{2(-i)(2k-{}^7/_3)}{3(3k-1)3k}$$

$$\frac{a_k''}{a_{k-1}''} = \frac{2(-i)(2k-{}^5/_3)}{3\cdot 3k(3k+1)} \qquad (k = 1, 2, \ldots)$$

As can be readily shown, this equation can be satisfied by assuming that

$$a_k = \frac{(-i)^k (2k)!}{(3k+2)!}, \qquad a_k' = \frac{(-i)^k \Gamma(2k-{}^1/_3)}{(3k)!}, \qquad a_k'' = \frac{(-i)^{k+1}\Gamma(2k+{}^1/_3)}{(3k+1)!}$$

$$(k = 0, 1, 2, \ldots)$$

Consequently, the desired functions w_0, w_1, w_2 and the general solution w of equation (2.4) equal

$$w_0 = C \sum_{k=0}^{\infty} \frac{(-i)^k (2k)!}{(3k+2)!} z^{3k+2}, \qquad w_1 = \sum_{k=0}^{\infty} \frac{(-i)^k \Gamma(2k-{}^1/_3)}{(3k)!} z^{3k}$$

$$w_2 = \sum_{k=0}^{\infty} \frac{(-i)^{k+1}\Gamma(2k+{}^1/_3)}{(3k+1)!} z^{3k+1}, \qquad w = w_0 + C_1 w_1 + C_2 w_2 \tag{3.1}$$

The series in (3.1) converge for all $z \neq \infty$, i.e., w is an integral function.

Since the desired, particular solution satisfies the condition Imw = 0 for z = iη, the arbitrary constants C_1, C_2 - as well as C - must be real.

In order to determine them, by first substituting the variables

$$\tau = -{}^4/_{27}\, iz^3, \qquad \arg\tau = 3\arg z + {}^3/_2\,\pi \tag{3.2}$$

we reduce equation (2.4) to a non-homogeneous confluent hypergeometric /58 equation

$$\tau\frac{d^2w}{d\tau^2} + \left(\frac{2}{3} - \tau\right)\frac{dw}{d\tau} + \frac{w}{6} = -\frac{C}{2^{4/3}\tau^{1/3}} \tag{3.3}$$

We can use the confluent, hypergeometric functions (Ref. 2, 3)

$$\Phi = \Phi(-{}^1/_6, {}^2/_3; \tau), \qquad \Psi = \Psi(-{}^1/_6, {}^2/_3; \tau) \tag{3.4}$$

as the linearly-independent particular solutions of the homogeneous equation corresponding to (3.3).

The Wronskian of the solution of (3.4) equals (Ref. 2)

$$W = \Phi\Psi_\tau{}' - \Psi\Phi_\tau{}' = -[\Gamma({}^2/_3)/\Gamma(-{}^1/_6)]\, e^\tau \tau^{-2/3} \tag{3.5}$$

We can write the solution of the non-homogeneous equation (3.3) by the method of variation of arbitrary constants, assuming that

$$w = u\Phi + v\Psi, \qquad w_\tau{}' = u\Phi_\tau{}' + v\Psi_\tau{}' \tag{3.6}$$

As is customary in the method of variation of constants, for the function u,v we obtain the equation

$$\frac{du}{d\tau} = \frac{C\Psi}{2^{4/3}\tau^{4/3}W} = -De^{-\tau}\tau^{-2/3}\Psi$$

$$\frac{dv}{d\tau} = De^{-\tau}\tau^{-2/3}\Phi, \qquad D = \frac{C\Gamma(-{}^1/_6)}{2^{4/3}\Gamma({}^2/_3)} \tag{3.7}$$

Formula (3.5) is employed to derive the relationship (3.7).

In view of (3.2), we have the following in the region of flow

$$\xi > 0,\ \eta < 0; \quad -{}^1/_2\pi < \arg z < 0; \quad 0 < \arg\tau < {}^3/_2\pi$$

Let us find the asymptotic behavior of the solution w for $\tau \to \infty$ in the sector $0 < \arg\tau < {}^1/_2\pi$. The following asymptotic formulas (Ref. 2) are valid in the given sector:

$$\Phi \sim [\Gamma({}^2/_3)/\Gamma(-{}^1/_6)]\, e^{\tau}\tau^{-5/6}, \quad \Psi \sim \tau^{1/6} \tag{3.8}$$

Let us substitute (3.8) in (3.7), and let us determine u,v - carrying out integration asymptotically. Then let us substitute u,v in expression (3.6) for w. We obtain

$$\begin{gathered} u \sim u(\infty) + De^{-\tau}\tau^{-1/2}, \qquad v \sim v(\infty) - \frac{2D\Gamma({}^2/_3)}{\Gamma(-{}^1/_6)}\tau^{-1/3} \\ w \sim -2D\frac{\Gamma({}^2/_3)}{\Gamma(-{}^1/_6)}\tau^{-1/3} + u(\infty)\Phi + v(\infty)\Psi + O(\tau^{-4/3}) \end{gathered} \tag{3.9}$$

In order that w have a given asymptotic behavior $w = O(z^{-1}) = O(\tau^{-1/3})$ for $\tau \to \infty$, it is necessary to equate the constants $u(\infty) = v(\infty) = 0$ to zero. Then, taking into account the value of (3.7) by the constant D and the relationship (3.2), from (3.9) we find the asmyptotic behavior

$$w \sim (3iC)/(2z) \quad \text{for } z \to \infty \tag{3.10}$$

The asymptotic behavior (3.10) is valid in the remaining portion of the flow region, which can be examined in a similar manner.

We obtained the following for the functions u,v from (3.7):

$$u(\tau) = -D\int_{\infty}^{\tau} e^{-\tau}\tau^{-2/3}\Psi\, d\tau, \qquad v(\tau) = D\int_{\infty}^{\tau} e^{-\tau}\tau^{-2/3}\Phi\, d\tau \tag{3.11}$$

The integration paths in (3.11) are initiated for $\tau \to \infty$, $|\arg\tau| < {}^1/_2\pi$.

The desired solution w can be determined unambiguously by the formulas (3.6), (3.4), (3.11). Substituting the known (Ref. 2) series expansions of the confluent hypergeometric functions (3.4) in powers of τ in formulas (3.11) and (3.6), we can determine the expansion of the function w for small τ /59

$$w = \left[u(0) + v(0)\frac{\Gamma({}^1/_3)}{\Gamma({}^1/_6)}\right] + v(0)\frac{\Gamma(-{}^1/_3)}{\Gamma(-{}^1/_6)}\tau^{1/3} + \frac{3D\Gamma(-{}^1/_3)}{2\Gamma(-{}^1/_6)}\tau^{2/3} + O(\tau)$$

In view of (3.11), the constants u(0), v(0) represent definite integrals along the real semi-axis (from 0 to ∞). These integrals are calculated on the basis of formulas given on pages 269-270 of the book (Ref. 2), or on page 874 of the book (Ref. 3). After simple transformations utilizing functional relationships for the gamma-function and the relationship (3.2), we finally obtain

$$w = {}^1/_2\, C\Gamma(-{}^1/_3) + {}^1/_2\, iC\Gamma({}^1/_3)\, z + {}^1/_2\, Cz^2 + O(z^3)$$

Comparing this expansion with formulas (3.1), we can find the constants: $C_1 = {}^1/_2\, C$, $C_2 = -{}^1/_2 C$.

In view of (3.1), the desired solution of equation (2.4) is

$$w(z) = \frac{C}{2}\sum_{k=0}^{\infty}\left\{(-i)^k z^{3k}\left[\frac{\Gamma(2k-{}^1/_3)}{(3k)!} + \frac{i\Gamma(2k+{}^1/_3)}{(3k+1)!}z + \frac{2(2k)!}{(3k+2)!}z^2\right]\right\} \quad (3.12)$$

The asymptotic behavior of the solution for (3.12) for $z \to \infty$ is determined by the formula (3.10). Employing the asymptotic series for the confluent hypergeometric function (Ref. 2), we can readily determine the asymptotic series for w. Let us present the final result which - just as formulas (3.1) - can be verified by direct substitution in equation (2.4):

$$w(z) \sim \frac{3iC}{2} \sum_{k=0}^{\infty} \frac{(-i)^k (3k)!}{(2k+1)!\, z^{3k+1}} \quad \text{for} \quad z \to \infty \tag{3.13}$$

Thus, the desired solution of equation (2.4) with the required asymptotic behavior can be determined in the form of a series (3.12), which converges for all finite z, in the form of an asymptotic series (3.13), and can also be determined by formulas (3.6), (3.11) by means of the confluent hypergeometric functions (3.4).

4. In order to make a definitive determination of the flow and form of a free surface, it is necessary to determine the constant C. First of all, from formula (3.12) we have:

$$w(0) = \varphi(0, 0) = {}^1/_2\, C\Gamma(-{}^1/_3), \qquad w'(0) = i\psi_\xi(0, 0) = {}^1/_2\, iC\Gamma({}^1/_3) \tag{4.1}$$

Let us introduce the additional function

$$P(\xi) = \frac{3}{2C} \int_0^\xi \frac{\psi_\xi(0, 0) - \psi_\xi(x, 0)}{x^2}\, dx \tag{4.2}$$

Employing equation (4.1), we can rewrite (4.2) in the form

$$P(\xi) = P(\infty) - \frac{3\Gamma({}^1/_3)}{4\xi} + \frac{3}{2C} \int_\xi^\infty \frac{\psi_\xi(x, 0)}{x^2}\, dx \tag{4.3}$$

For the function $\psi_\xi(\xi,0) = \operatorname{Im} w'(\xi)$, we can readily obtain both the convergent and the asymptotic series from formulas (3.12) and (3.13). /60

Substituting the first of these series in (4.2), and the second - in (4.3), we can determine the convergent and asymptotic (for $\xi \to \infty$) series for P:

$$
\begin{aligned}
P(\xi) &= \frac{3}{4}\sum_{k=0}^{\infty}\left\{(-1)^k \xi^{6k+1}\left[\frac{\Gamma(4k+{}^5/_3)}{(6k+1)(6k+2)!} + \right.\right.\\
&\left.\left. + \frac{2(4k+2)!\,\xi^2}{(6k+3)(6k+4)!} + \frac{\Gamma(4k+{}^{13}/_3)\,\xi^4}{(6k+5)(6k+6)!}\right]\right\}\\
P(\xi) &\sim P(\infty) - \frac{3\Gamma({}^1/_3)}{4\xi} - \frac{3}{4}\sum_{k=0}^{\infty}\frac{(-1)^k(6k+1)!}{(2k+1)(4k+1)!\,\xi^{6k+3}} =\\
&= P(\infty) - \frac{3\Gamma({}^1/_3)}{4\xi} - \frac{3}{4\xi^3} + O(\xi^{-9}) \quad (\xi \to \infty)
\end{aligned}
\tag{4.4}
$$

Comparing formulas (2.5), (2.6) and (4.2), (4.3), we can express the constant C and the form of the free surface $f(\xi)$ by the function $P(\xi)$

$$
C = \frac{\varepsilon}{P(\infty)}, \qquad f(\xi) = \varepsilon\left\{-\frac{3\Gamma({}^1/_3)}{4P(\infty)} + \xi\left[1 - \frac{P(\xi)}{P(\infty)}\right]\right\} \tag{4.5}
$$

The determination of the constant C, which is included with the multiplier in formulas (3.12), (3.13) for w, as well as the determination of the function $f(\xi)$, can be reduced (as can be seen from [4.5]) to calculating the function $P(\xi)$ and, in particular, $P(\infty)$. The series (4.4) can be utilized for this purpose.

Another method employed in this work is as follows. Let us examine equation (2.4) on the real axis, assuming $z = \xi$, $w = \varphi + i\psi = C(y_1 + iy_2)$ in it. By separating the real and imaginary parts in (2.4), we can obtain a system of two equations of the second order for the functions $y_1(\xi)$, $y_2(\xi)$

$$
y_1'' = ({}^4/_9)\,\xi^2 y_2' - ({}^2/_9)\,\xi y_2 + 1, \qquad y_2'' = ({}^2/_9)\,\xi y_1 - ({}^4/_9)\,\xi^2 y_1'.
$$

This system was integrated numerically on a EVM (computer) from $\xi = 0$ to $\xi = 20$ for the initial data which were obtained from (3.12)

$$y_1 = {}^1/_2\,\Gamma\,(-{}^1/_3), \quad y_1' = y_2 = 0, \quad y_2' = {}^1/_2\,\Gamma\,({}^1/_3) \quad \text{for} \quad \xi = 0$$

The function $P(\xi)$ was determined by means of y_2 by quadrature of (4.2)

$$P(\xi) = \frac{3}{2}\int_0^{\xi} \frac{y_2'(0) - y_2'(x)}{x^2}\,dx = \frac{3}{4}\int_0^{\xi} \frac{\Gamma({}^1/_3) - 2y_2'(x)}{x^2}\,dx \qquad (4.6)$$

The indeterminate form of the integrand in (4.6) can be readily expanded for $x = 0$, and for small ξ we have the following from (4.4)

$$P(\xi) = {}^3/_8\,\Gamma({}^5/_3)\,\xi + O(\xi^3)$$

Determination of $P(\xi)$ was controlled by calculating the convergent series (4.4).

The value of $P(\infty)$ was calculated by $P(\xi)$ for $\xi = 15 \div 20$, utilizing the latter from formula (4.4). The function $f(\xi)$ was determined by means of $P(\xi)$ according to formula (4.5).

Let us present some computational results, in which (2.1), (4.1), (4.5) were also used:

$$P(\infty) = 2.356, \qquad C = \varepsilon / P(\infty) = 0.4244\varepsilon$$
$$f(0) = -{}^3/_4\,\varepsilon\Gamma({}^1/_3) / P(\infty) = -0.8527\varepsilon, \quad f'(0) = \varepsilon$$
$$f''(0) = {}^1/_3\,\varphi(0, 0) = {}^1/_6\,\varepsilon\Gamma(-{}^1/_3) / P(\infty) = -0.2874\varepsilon$$
$$V = |w'(0)| = {}^1/_2\,\varepsilon\Gamma({}^1/_3) / P(\infty) = -{}^2/_3 f(0) = 0.5685\,\varepsilon$$

Here, V is the modulus of the dimensionless liquid velocity at /61 the origin (the velocity is directed along the η-axis). Figure 2 presents a graph of the function $\varepsilon^{-1}f(\xi)$ (of the free liquid surface). The graph clearly shows oscillations, whose frequency increases, and the amplitude rapidly decreases with an increase in ξ. These oscillations correspond to capillary waves which are propagated along the free liquid surface according to a self-similar law. The shorter waves are propagated with a larger

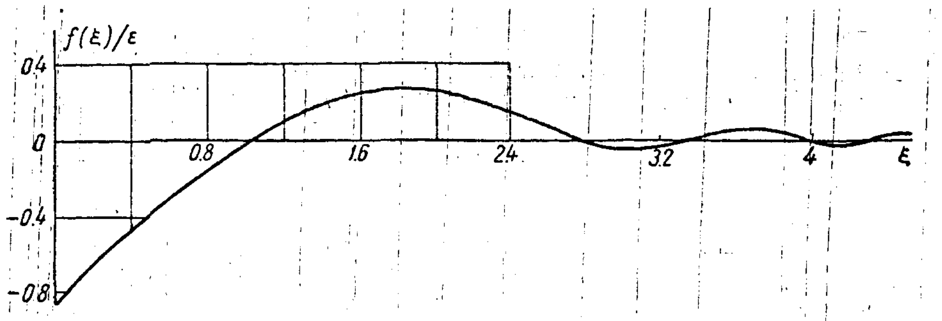

Figure 2

velocity, in accordance with the general characteristics of capillary waves (Ref. 1). In the case of $\xi \gg 1$, the function $\varepsilon^{-1}f(\xi)$ has asymptotic behavior, which follows from (4.4) and (4.5)

$$\varepsilon^{-1} f(\xi) \sim {}^3/_4 \, [P(\infty)\, \xi^2]^{-1}$$

and strives to zero, remaining positive.

Thus, the linearized self-similar problem has been completely solved for $\alpha = {}^1/_2\pi$. We must remember that $\varepsilon = \gamma - {}^1/_2\pi$, and therefore $\varepsilon > 0$ for a non-wetting liquid and $\varepsilon < 0$ for a wetting liquid. The change to dimensional variables is given by formulas (1.6) - for example, the rising of the liquid and its velocity at the wall equal

$$f^\circ(0, t) = \left(\frac{\sigma t^2}{\rho}\right)^{1/3} f(0) = -0.8527\left(\gamma - \frac{\pi}{2}\right)\left(\frac{\sigma t^2}{\rho}\right)^{1/3}$$

$$v^\circ = \frac{\partial f^\circ(0, t)}{\partial t} = \frac{2}{3}\left(\frac{\sigma}{\rho t}\right)^{1/3} f(0) = -0.5685\left(\gamma - \frac{\pi}{2}\right)\left(\frac{\sigma}{\rho t}\right)^{1/3}$$

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